2022 AIME I Problem 4

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LICENSE

Problem Link

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Commentary

In this problem, $w = e^{\pi i/6}$ and $z = e^{5\pi i/3}$. Thus, we have:

$$i \cdot w^r = z^s \implies e^{\pi i/2} \cdot e^{\pi i r/6} = e^{5\pi i s/3}$$

from the condition given in the problem.

I messed up on the test by assuming this meant

$$\frac{\pi i}{2} + \frac{\pi i r}{6} = \frac{5\pi i s}{3}$$

instead of

$$\frac{\pi i}{2} + \frac{\pi i r}{6} = \frac{5\pi i s}{3} + 2\pi i k.$$

Now, when I solved the second equation, I simplified it:

$$\begin{aligned} \frac{\pi i}{2} + \frac{\pi i r}{6} &= \frac{5\pi i s}{3} + 2\pi i k \\ 3\pi i + \pi i r &= 10\pi i s + 12\pi i k \\ 3 + r &= 10s + 12k \\ 3 + r - 10s &= 12k \\ r + 2s &\equiv 9 \pmod{12} \end{aligned}$$

I then noticed that r must be $1, 3, 5, 7, 9$ or 11 mod 12, and I found the conditions on the values of s for each case.

When $r = 9$, I got that $s \equiv 0, 6 \pmod{12}$ and when $s \equiv 0 \pmod{12}$ I counted 9 possible ways, when there were only 8 possible ways.

In the end, after fixing this, I got the correct answer.

This casework is kind of long, so I may try to find another way.